The shortest pure kinematic study of the Foucault pendulum, with geometric algebra.

We will use following systems of coordinates in the Northern Hemisphere :

– (S1) is a quasi-inertial system whose center coincides with the Earthcenter, orthonormal, right handed, whose axis 3 (unit vector w) is directed towards the Northpole, and whose other axes remain fixed relatively to the stars ; that means that we neglect principally the fact that the Earth rotates around the Sun ;

– (S2) is an Earthbound orthonormal coordinate system centered at the same origin, whose axis 3 is oriented towards the point where the pendulum is suspended, which we name x (x is a geometric algebra vector, unit vector $\hat{x}$, $x = R\hat{x}$) ; axis 1 with unit vector u is southward oriented in the plane defined by x and the Earth-axis ; thus axis 2 with unit vector v is always perpendicular to that plane and eastward oriented ; we can write :

(1) $\quad w = \hat{x}\sin\lambda - u\cos\lambda \qquad I = uv\hat{x} \qquad I\hat{x} = uv \qquad Iu = v\hat{x} \qquad Iv = \hat{x}u$

where I represents the pseudoscalar in $\mathcal{G}^3$, geometric algebra associated to $\mathcal{R}^3$, and λ is the latitude angle.

– (S'2) is the (S2) coordinate system translated to the surface of the Earth to materialize in the marble table on which the pendulum plane's rotational move can be observed ; we will not need it for our calculations, but it is important to note that the trace of the pendulum plane observed on that horizontal table is parallel to the trace of the same plane which we would observe, if possible, on the (u, v) plane in (S2) ;

The bob of the pendulum is designated by the vector y . Thus the plane of the pendulum is defined by the origin O center of Earth, and the vectors x and y . We have of course always $|y - x| = l$.

One critical question in the Foucault pendulum theory is the fact that the plane[1] in which the pendulum oscillates should remain fixed in (S1), that is relatively to the stars, which is of course impossible because the suspension vector x moves with the Earth's rotation.

But let us look at two infinitesimally proximate instants t and $t + \delta t$. During that time the rotation movement transporting the pendulum can safely be approximated[2] by a translation movement with vector :

(2) $\delta x = x.(wI)\delta\theta = I\hat{x}\wedge wR\delta\theta = v\cos\lambda R\delta\theta \qquad$ and $\quad \delta\theta = \omega\delta t$,

where ω is the rotational speed of the Earth around the w axis. Thus we can admit that during that short time span the relative speed vector of the pendulum is parallel transported in $\mathcal{R}^3$, that is without being constraint by the curvature of the Earth. Of course we observe the result of that movement in the displaced (S'2) system, whose rotation we then cannot neglect[3] .

Let us now consider in (S'2) the allways southward oriented vector $u(t)$ which we can engrave in the earthbound (u, v) table. At the instant $(t + \delta t)$ that vector will be moved –rotated in (S1) – to the position :

(3) $u(t + \delta t) = u(t) + u.(Iw)\delta\theta = u + Iu\wedge w\delta\theta = u + v\sin\lambda\delta\theta$

But if we project the parallel transported vector $u(t)$ on the new horizontal (S'2) table we get :

(4) $Pu = u\wedge(\hat{x} + \delta\hat{x})(\hat{x} + \delta\hat{x}) = u - u.(\hat{x} + \delta\hat{x})(\hat{x} + \delta\hat{x}) = u - 0 = u \qquad \delta u = 0$

That means that the engraved u (our reference mark) has moved – in (S1) – by a small angle towards the East (a counterclockwise move), whereas the projection of the infinitesimally parallel transported u has not directionally moved - in (S1). Thus if somebody placed in (S'2) had observed the moves of a pendulum precisely South swinging at instant t , he would have noticed that the pendulum plane had moved by an angle $\sin\lambda\delta\theta$ westwards during the δt time interval (a clockwise move).

1. Of course we do not study here the general movements of spherical pendulums. We suppose that the speed vector of the bob relative to (S2) remains always in the oscillation plane.

2. The error induced by that approximation between δx and δy is of order 10^{-6} .

3. Mathematically that corresponds to the definition of parallel displacement on a manifold. We must project on the tangent plane.

What happens if we project the parallel transported vector v ? We simply get :

(5) $Pv = v \wedge (\hat{x} + \delta\,\hat{x})\,(\hat{x} + \delta\,\hat{x}) = v - v\,.(\hat{x} + \delta\,\hat{x})\,(\hat{x} + \delta\,\hat{x}) = v - v.\delta\hat{x}\,(\hat{x} + \delta\,\hat{x}) = v - \hat{x}\cos\lambda\,\delta\theta$

and we have $v.Pv = 1$ Thus[4] we would observe, in (S'2) by comparison with the engraved South direction, the same clockwise $\sin\lambda\,\delta\theta$ move of a pendulum swinging in the v direction.

Of course for every intermediate swinging direction the combined u and v moves would produce the same $\sin\lambda\,\delta\theta$ clockwise rotation. As those moves can be added, that is integrated in (S'2), ***we have proved the Foucault pendulum propriety, with pure kinematical arguments.***

Indeed that could be resumed in two phrases. To explain the behaviour of the Foucault pendulum there is no need to introduce and analyze mysterious fictitious forces ; one must only observe that during a time interval as short as you like, the swinging plane's direction can be considered fixed[5] relatively to the stars, but its trace on the observation table moves because that table rotates under the pendulum with the constant angular speed $\omega\sin\lambda$ around the instantaneous mobile $\hat{x}(t)$ axis.[6]

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4. More precisely if we call $(\pi/2 - \varphi)$ the angle between the engraved South mark and Pv we get :

$\sin\varphi = (u + v\,\sin\lambda\,\delta\theta).(v - \hat{x}\cos\lambda\,\delta\theta) = \sin\lambda\,\delta\theta \quad \Longrightarrow \quad \varphi = \sin\lambda\,\delta\theta$ which was to be proved.

5. One should note that the relative speed of the bob is approximately 100 times slower than the quasi-translational move of the whole system. That ensures that the speed vector of the bob is indeed parallel transported in (S1).

6. A more precise calculation shows us that the infinitesimal angle variation $\sin\lambda\,\delta\theta$ is valid up to order $(\delta\theta^3)$, which justifies if necessary the validity of the finite variation.